Group Theory-Week 2

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Contents

1 Cosets **2**
1.1 Definition . . . 2
1.2 Properties of Cosets . . . 2
1.3 Lagrange's Theorem . . . 2
1.4 Examples . . . 2

2 Applications **3**
2.1 Order of an element . . . 3
2.2 p^{th} roots of unity . . . 3

3 Exercises **3**

1 Cosets

1.1 Definition

Let G be a group and H be its subgroup. Let a be any element of G.
We define

$$aH = \{a \cdot h \mid h \in H\}$$

Note that aH is a set of elements and is called a left coset of H corresponding to a.

1.2 Properties of Cosets

Lemma 1 $a \in aH$

Use the fact that identity belongs to any subgroup

Lemma 2 If $b \in aH$ then $aH = bH$

Let $b = ah_1$. So for any $h \in H, bh = a(h_1h) \in aH$ implying $bH \subset aH$. As $a = bh_1^{-1}, a \in bH$, apply same logic to get the result.

Lemma 3 The function $f : H \to aH$ given by $f(h) = ah$ is a bijection

The function is surjective by definition, for injective if $f(h_1) = f(h_2)$, then $ah_1 = ah_2$ now multiply on left by inverse of a

Lemma 4 If G is finite, all left cosets have same size

Directly follows from Lemma 3

Lemma 5 If aH, bH are two left cosets then they are either disjoint or equal.

If they are not disjoint, there exists $h_1, h_2 \in H$ such that $ah_1 = bh_2 \implies a(h_1h_2^{-1}) = b \implies b \in aH$, now use Lemma 2

This means every element of G is in exactly one left coset with respect to H. Hence the left cosets of G with respect to H partition G into disjoint sets $S_1, S_2, \cdots, S_t$ where each $S_i = aH$ for some (every) $a \in S_i$. And moreover the size of each S_i is same.

1.3 Lagrange's Theorem

If G is a finite group of size n and H a subgroup of size m then m divides n

Proof - Let the left cosets of G be $S_1, S_2, \cdots, S_t$. Let identity e be in S_k, then we have $S_k = eH = H$. In particular the size of each S_i is m. As S_i form a partition of the group $n = mt$ and hence m divides n

1.4 Examples

Exhibit the left cosets of $\mathbb{Z}$ with respect to the subgroup $3\mathbb{Z}$. (Group operation addition)
The left coset of $3\mathbb{Z}$ of '0' is $0+3\mathbb{Z} = \{..., -6, -3, 0, 3, 6, ...\}$
The left coset of $3\mathbb{Z}$ of '1' is $1+3\mathbb{Z} = \{..., -5, -2, 1, 4, 7, ...\}$,
The left coset of $3\mathbb{Z}$ of '2' is $2+3\mathbb{Z} = \{..., -4, -1, 2, 5, 8, ...\}$.
Now, the left coset of $3\mathbb{Z}$ of '3' is $3+3\mathbb{Z}$ = again $\{..., -6, -3, 0, 3, 6, ...\}$
So, the left cosets repeat , so we take the distinct left cosets from all the left cosets. Hence the left cosets of $3\mathbb{Z}$ of $\mathbb{Z}$ are $\{..., -6, -3, 0, 3, 6, ...\}$ $\{..., -5, -2, 1, 4, 7, ...\}$ $\{..., -4, -1, 2, 5, 8, ...\}$.

2 Applications

2.1 Order of an element

Consider a finite group G and any element a.
Consider the elements $e = a^0, a, a^2, a^3, \cdots,$ as G only has finitely many elements there must be a repetition, say $a^u = a^v$, this implies $a^{u-v} = e$ (assuming $u > v$)
The smallest positive integer u such that $a^u = e$ is called the order of a.
Notice that $\{e, a, a^2, \cdots, a^{u-1}\}$ is a subgroup of G of size u. In particular u divides the size of G

2.2 p^{th} roots of unity

Firstly, by Fundamental Theorem of Algebra $x^p - 1$ has exactly p roots counting multiplicity, moreover by taking derivative one can check that there are no repeated roots and hence it has exactly p distinct roots.
If a, b are roots of $x^p = 1$ then it is easy to see that so are ab, a^{-1}.
Moreover 1 is a root of the equation. Hence the roots form a group under multiplication operation.
Consider any root u other than 1, then order of u is > 1 but divides the size of the group which is p.
Hence order of u must be p and $\{1, u, \cdots, u^{p-1}\}$ are distinct elements.
But $x^p - 1$ has only p roots implying $\{1, u, \cdots, u^{p-1}\}$ are all the roots of unity.

3 Exercises

1. A group G is cyclic if there is an element in G such that the subgroup generated by that element is the entirety of G. Prove that any group with prime number of elements is cyclic.

2. Is it be possible that H is finite but G has infinitely many left cosets with respect to H?

Reference for week 2 : Section 10 from fraleigh book